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MODELLING OF DYNAMIC SYSTEMS IN STATE SPACE

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Abstract: This paper deals with the solution of dynamical systems in state space. Complicated differential equations are converted into a simpler form by using state variables in vector matrix. It is used for multi-input and multi-output systems, and the solution is performed using matrix notation. It describes systems with complex internal structure. It allows state models to be manipulated using matrix calculus. Systems described by a state model are characterized by the fact that it is easier to design state control for them.

1 Introduction

Classical control theory and the methods we have used so far are based on a simple description of the input and output of a system, usually expressed as a transfer function. These methods use no information about the internal structure of the device and are limited to systems with one input and one output, where we have seen only limited control of closed-loop behaviour using feedback control.

Modern control theory solves many of the constraints using a much richer description of the dynamics of the devices. The trend in engineering systems is towards greater task complexity, especially due to the requirements of complex tasks and good accuracy. Complex systems may have multiple inputs, multiple outputs, and may be time-varying. Due to the need to meet increasingly stringent performance requirements of control systems, the increase in system complexity, and easy access to computers, modern control theories are an approach to the analysis and design of complex control systems. This new approach is based on the concept of state. The state concept itself is not new, as it has been around for a long time in classical dynamics and other fields [1-7].

2 State space representation

A model is a mathematical representation of a physical, biological or information system. Models allow us to predict how a system will behave. In this text we will be interested in models in the so-called state form, where phenomena do not happen instantaneously, e.g., the speed of a car does not change instantly when the pedal is pressed, nor does the temperature in a room change instantly when the air conditioning is turned on.

In corporate systems, increasing research funding for a project will not increase returns in the short term but may increase them in the long term (if it is a good investment). These are all examples of dynamic systems whose behaviour changes with time. Another perspective on dynamics comes from electrical engineering. The prototype of such a problem was the description of electronic amplifiers. It was natural to view an amplifier as a device that transforms input voltages into output voltages, neglecting the internal details of the amplifier.

This resulted in an input-output view of systems. Dynamic systems can be viewed in two ways: an internal and an external view. The internal view attempts to describe internal regularities and comes from classical mechanics. The prototype of such a system was the description of the motion of the planets. For this problem it was natural to give an overall characterization of the motion of all the planets. This requires a rigorous analysis of the action, gravitational action, and relative positions of the planets in the system. The two different views were merged into a control theory. Models based on the internal view are called internal descriptions, state models, or white box models. Models based on the external view are called external descriptions, input-output models, or black box models.

2.1 State variables

The state variables of a dynamical system are the variables forming the smallest set of variables that determine the state of the dynamical system. If at least n variables x_1, x_2, ..., x_n are needed to completely describe the behavior of a dynamical system such that once the input is given for $t \geq t_0$ and the initial state at time $t = t_0$ is specified, the future state of the system is completely determined, then such n variable set is a state variable.

State variables need not be physically measurable or observable quantities. Variables that do not represent physical quantities that are neither measurable nor observable can be selected as state variables. Such freedom in the selection of state variables is an advantage of state methods. In practice, however, it is convenient to choose an easily measurable quantity on the state variables, if at all possible, because optimal control of the control laws will require feedback from all state variables with a suitable value.

2.2 State equation

In state space analysis, we are concerned with three types of variables that are involved in the modelling of dynamical systems: input variables, output variables, and state variables. The state system representation for a given system is not unambiguous, except that the number of state variables is the same for any of the different state representations of the same system.

Suppose also that there are r inputs $u_1(t)$, $u_2(t)$, ..., $u_r(t)$ and m outputs $y_1(t)$, $y_2(t)$,..., $y_m(t)$. We define the n outputs of the integrators as state variables: $x_1(t)$, $x_2(t)$,..., $x_n(t)$.
The system can be described by the equations:

$$\begin{aligned}\dot{x}_1(t) &= f_1(x_1, x_2, \dots, x_n;\ u_1, u_2, \dots, u_r;\ t)\\ \dot{x}_2(t) &= f_2(x_1, x_2, \dots, x_n;\ u_1, u_2, \dots, u_r;\ t)\\ \dot{x}_n(t) &= f_n(x_1, x_2, \dots, x_n;\ u_1, u_2, \dots, u_r;\ t)\end{aligned} \tag{1}$$

Outputs $y_1(t)$, $y_2(t)$, ..., $y_m(t)$ are defined as:

$$\begin{aligned}y_1(t) &= g_1(x_1, x_2, \dots, x_n;\ u_1, u_2, \dots, u_r;\ t)\\ y_2(t) &= g_2(x_1, x_2, \dots, x_n;\ u_1, u_2, \dots, u_r;\ t)\\ y_m(t) &= g_m(x_1, x_2, \dots, x_n;\ u_1, u_2, \dots, u_r;\ t)\end{aligned} \tag{2}$$

Then we write equations (1) and (2) :

$$\dot{x}(t) = f(x, u, t) \tag{3}$$

$$y(t) = g(x, u, t) \tag{4}$$

where equation (3) is the state equation and equation (4) is the output equation. The vector functions f and g involve time t and such a system is called a time-varying system.

If equations (3) and (4) are linearized about the operating state, then they have the following linearized state equations and output equations:

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t)\\ y(t) &= C(t)x(t) + D(t)u(t)\end{aligned} \tag{5}$$

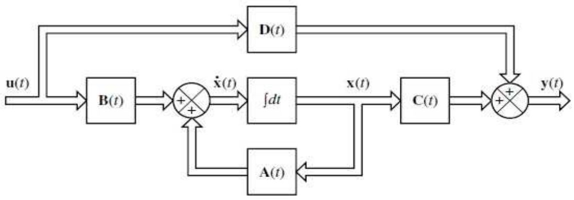

Figure 1 Graphical representation of state equations [1]

Matrix:
A (t) system matrix,
B (t) input matrix,
C (t) output matrix,
D (t) feedforward matrix.

3 Computer simulation of dynamic systems

A simple mechanical oscillator will be used as a first example to express the state description (Figure 2). Consider a mechanical system consisting of a body of mass m fixed by a spring of stiffness k against a rigid frame.

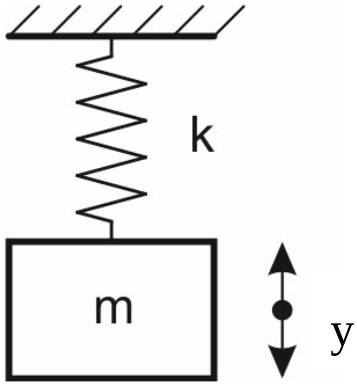

Figure 2 Mechanical oscillator

The equation of motion of the system is:

$$m\ddot{y}(t) + ky(t) = 0 \tag{6}$$

We define the state variables $x_1(t)$ and $x_2(t)$ as:

$$\begin{aligned}x_1 &= y\\ x_2 &= \dot{y}\end{aligned} \tag{7}$$

If we perform the derivation of the state variables, then it is possible to write:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\frac{k}{m}x_1\end{aligned} \tag{8}$$

The dynamic system is not excited by an external force, the initial condition is given to the initial condition x_{10}=0.3m.

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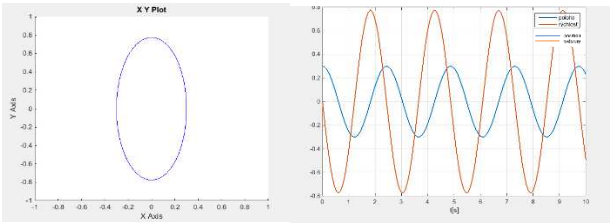

Figure 3 State trajectory and plot of results kinematic parameters

Consider the mechanical system p in Figure 4, which consists of a mass m, a spring k and a damper b on which a force f(t) acts. We assume that the system is linear.

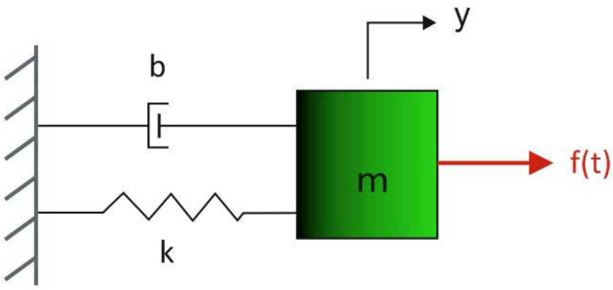

Figure 4 Second-order dynamic system

The external force f(t) is the input to the system and the displacement of mass y is the output. This system is a system with one input and one output and has one degree of freedom. The equation of motion of the system is:

$$m\ddot{y} = f(t) - f_b - f_k$$
$$m\ddot{y} = f(t) - b\dot{y} - ky$$
$$m\ddot{y} + b\dot{y} + ky = f(t) \tag{9}$$

The second-order differential equations describes the system, then two simultaneous, first-order differential equations are required along with two state variables. We define the state variables $x_1(t)$ and $x_2(t)$ as

$$x_1(t) = y(t)$$
$$x_2(t) = \dot{x}_1(t) = \dot{y}(t) \tag{10}$$

Equation (10) can be solved for the state variables. For a linear, time-invariant, second order system with a single input, the state equations could take on the following form. Furthermore, it is possible to write:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 == -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}f \tag{11}$$

Where x1 and x2 are the state variables. There is a single output, the output equation cold take on the following form:

$$y = x_1 \tag{12}$$

The coice of state variables for a given system is not unique. The requirement in choosing the state variables is that they be liearly independent and that a minimum number of then be chosen.

In the standard form it is possible to write:

$$\dot{x} = \boldsymbol{Ax} + \boldsymbol{Bu}$$
$$\boldsymbol{y} = \boldsymbol{Cx} + \boldsymbol{Du}$$

In matrix form, equations (11) and (12) can be written:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{b}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix} f$$
$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \tag{13}$$

where

$$A = \begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{b}{m}\end{bmatrix} \quad B = \begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}$$
$$C = \begin{bmatrix}1 & 0\end{bmatrix} \quad D = 0 \tag{14}$$

Plot of results kinematic parameters such as the position and velocity of the mass is on Figure 5.

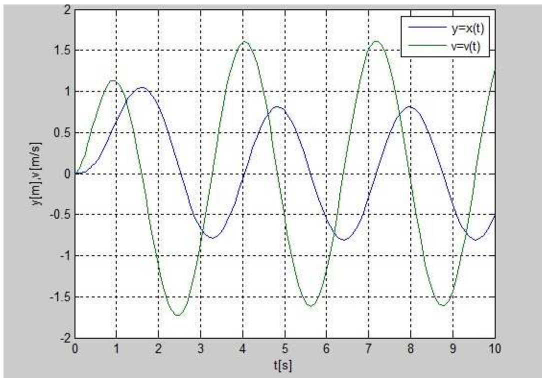

Figure 5 Representation of position x(t) and velocity v(t)

By examining a large number of dynamical systems, it was found that the shapes of the trajectories of the systems can take many different forms. Some of them are convergent, monotonous or periodic to some limit of what can be a point or a set of points (a circle). We are talking about a stable system. On the x-axis is represented the state variable x_1, i.e. the position, and on the y-axis the state variable x_2, i.e. the velocity of the mass m (Figure 6).

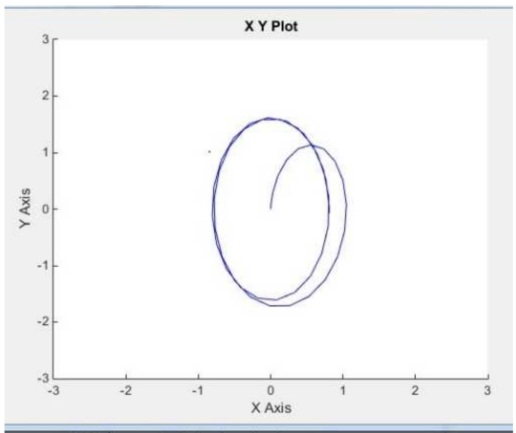

Figure 6 State trajectory of mechanical system

In the next case, consider the mechanical system shown in Figure 7, which consists of two bodies of masses m_1 and m_2 fixed in series by means of three springs of stiffness k_1, k_2 and k_3 on a rigid frame. An excitation force F acts on the body of mass m1 in the positive direction of the x_1 axis.

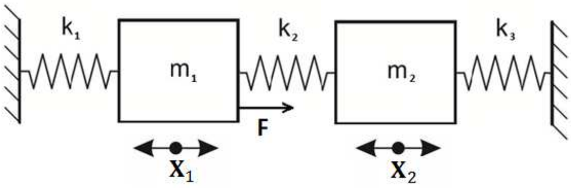

Figure 7 Two-mass dynamic system

In this case we get two equations of motion of the form:

$$m_1\ddot{y}_1 = -k_1 y_1 + k_2(y_2 - y_1) + F$$
$$m_2\ddot{y}_2 = -k_2(y_2 - y_1) + k_3 y_2 \qquad (15)$$

After modification we get the shape:

$$\ddot{y}_1 = (-k_1 y_1 + k_2(y_2 - y_1) + F)/m_1$$
$$\ddot{y}_2 = (-k_2(y_2 - y_1) + k_3 y_2)/m_2 \qquad (16)$$

Choosing state variables:

$$x_1 = y_1$$
$$x_2 = \dot{y}_1$$
$$x_3 = y_2$$
$$x_4 = \dot{y}_2 \qquad (17)$$

Derivations of state variables can be written:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{k_1 x_1}{m_1} + \frac{k_2(x_3 - x_1)}{m_1} + \frac{F}{m_1}$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = -\frac{k_2(x_3 - x_1)}{m_2} + \frac{k_3 x_3}{m_2} \qquad (18)$$

Equation of state of the system:

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{B}u \qquad (19)$$

Then we write the equations (18) in matrix form:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ -(k_1+k_2) & 0 & k_2 & 0\\ 0 & 0 & 0 & 1\\ k_2 & 0 & (k_2-k_1) & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ 1\\ 0\\ 0\end{bmatrix}F \qquad (20)$$

Plot of results kinematic parameters such as the position and velocity of individual masses can be seen on the Figure 8.

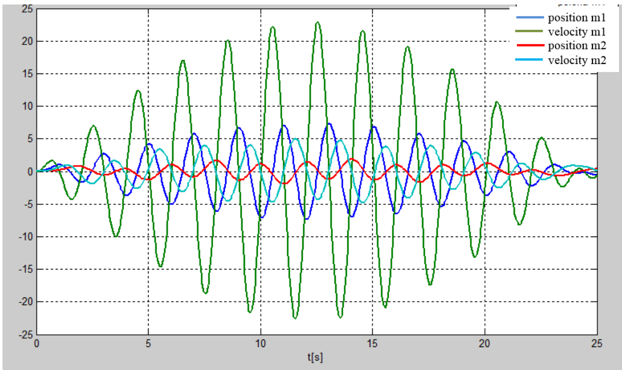

Figure 8 Position and velocity of the mechanical system in Matlab

4 Conclusions

The aim of the paper was to define a state space description of dynamical systems. and then apply it to examples of mechanical systems. Second-order differential equations were expressed and written in the form of vector matrices using state variables The state description of a dynamical system can describe systems with multiple inputs and outputs and systems with complex internal structure. The results of the solutions of the mechanical systems were obtained using Matlab/Simulink.

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