Acta Mechatronica - International Scientific Journal about Mechatronics
Volume: 1 2016 Issue: 2 Pages: 15-20 ISSN 2453-7306

THE CONTROL OF HOLONOMIC SYSTEM

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Keywords: double inverted pendulum, Lagrange function, linearization, state space, PID regulator
Abstract: This paper deals with the issue of mathematical modelling of the double inverted pendulum. The paper consists of the determination of mathematical model created via Lagrangian, the linearization of system and the design of linear quadratic regulator. For linear stable system were chosen DC motors placed to joints. Further for these motors were set individual components of PID regulator. The last part of article deals with simulation of double inverted pendulum.

1 Introduction

The study of humanoid robots is currently one of the most exciting research projects. Even if some of those works have already demonstrated very reliable dynamic biped walking (Yamaguchi, Soga, Inoue & Takanishi, 1999; Hirai, Hirose, Haikawa & Takenaka, 1998; Nishiwaki, Sugihara, Kagami, Kanehiro, Inaba & Inoue, 2000), we believe it is still important to understand the mathematical theoretical background of biped locomotion. The locomotion of human body can be considered as the movement of inverted pendulum with certain number of joints, e.g. we can consider human arm as triple inverted pendulum [1].

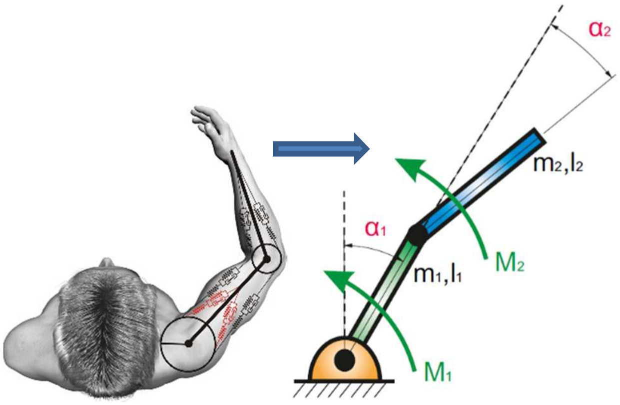

Figure 1 Human arm as double inverted pendulum

An inverted pendulum system is a typically nonlinear, redundancy, uncertainty, strong coupling and natural characteristics of instabilities. All these features make it the ideal model of advanced control theory and typical experiment platform of test control results. There are a number of different kinds of the inverted pendulum systems presenting a variety of control challenges. The most common types are [1]:

- the single inverted pendulum on a cart,
- the double inverted pendulum on a cart,
- the double inverted pendulum with an actuator at the first joint only,
- the double inverted pendulum with an actuator at the second joint only,
- the light weight rotary pendulum and
- other combinations.

In this paper was solved the stability problem of *the double inverted pendulum with actuators at both joints* in the upright position and at joints were used the same DC motors. The pendulum is pivoted at the lower end of inner arm (Figure 1).

The first step to achieve the objective is to understand the dynamics of the system of double inverted pendulum by developing the mathematical modelling of the system. In modelling, we have used Euler-Lagrange formulation to find equation of motion. In the second step, we linearized this non-linear system of double inverted pendulum in the up-up position and builded up its linear

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state space model. The linearization is one of the most important issues for control of non-linear systems. In the next step, the stability and controllability criteria showed that the system is unstable but it is controllable.

2 The motion equation of the double inverted pendulum

The position and orientation of the double inverted pendulum in the plane is represented by two *shape variables*–angles $q=(\alpha_1,\alpha_2)$. Then *the configuration space* of pendulum is $Q=G\times M=M=\alpha_1\times\alpha_2$. This configuration space can be visually represented as *a torus* T^2 that arises as combination of two basic building blocks of configuration space, i.e. by combining two circles (Figure 2) [2].

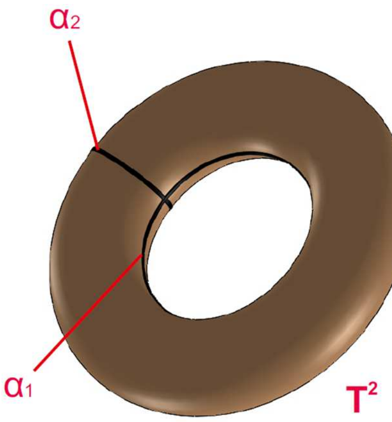

Figure 2 The configuration space of double inverted pendulum

The double inverted pendulum belongs to *holonomic systems*. For holonomic systems apply that their holonomic constraints remove degrees of freedom from a system, reducing the dimensionality of its configuration space. Formally, a holonomic constraint is defined as a (possibly time-varying) constraint function *f* on the system's configuration space *Q*. The *zero set* of the function forms the *accessible manifold* of the constrained system, the set of configurations satisfying the constraint [3].

The mathematical model of pendulum was derived using the Euler-Lagrange equation [4]:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}}-\frac{\partial L}{\partial \alpha}=M, \tag{1}$$

where $L=Ek\text{-}Ep$ is *Lagrangian*, E_k is *kinetic energy*, E_p is *potential energy* and *M* is generalized torque produced by actuators placed at joints. The kinetic energy of inverted pendulum has the form:

$$E_{k1}=\frac{1}{6}m_1 l_1^2\dot{\alpha}_1^2, \tag{2}$$

$$E_{k2}=\frac{1}{6}m_2 l_1^2\dot{\alpha}_1^2+\frac{1}{6}m_2 l_2^2\dot{\alpha}_2^2+ \frac{1}{3}m_2 l_1 l_2\dot{\alpha}_1\dot{\alpha}_2\cos(\alpha_1-\alpha_2). \tag{3}$$

The potential energy is defined as:

$$E_{p1}=m_1 g\frac{l_1}{2}\cos\alpha_1, \tag{4}$$

$$E_{p2}=m_2 g\left(l_1\cos\alpha_1+\frac{l_2}{2}\cos\alpha_2\right). \tag{5}$$

After substituting of individual terms of kinetic and potential energy to Lagrangian and after substituting to equation (1), we get the motion equations of double inverted pendulum:

$$\begin{aligned}&\frac{1}{3}m_1 l_1^2\ddot{\alpha}_1+\frac{1}{3}m_2 l_1^2\ddot{\alpha}_1+\frac{1}{3}m_2 l_1 l_2\ddot{\alpha}_2\cos(\alpha_1-\alpha_2)\\&-\frac{1}{3}m_2 l_1 l_2\dot{\alpha}_2\sin(\alpha_1-\alpha_2)(\dot{\alpha}_1-\dot{\alpha}_2)\\&+\frac{1}{3}m_2 l_1 l_2\dot{\alpha}_1\dot{\alpha}_2\sin(\alpha_1-\alpha_2)-m_1 g\frac{l_1}{2}\sin\alpha_1\\&-m_2 g l_1\sin\alpha_1=M_1,\end{aligned} \tag{6}$$

$$\begin{aligned}&\frac{1}{3}m_2 l_2^2\ddot{\alpha}_2+\frac{1}{3}m_2 l_1 l_2\ddot{\alpha}_1\cos(\alpha_1-\alpha_2)\\&-\frac{1}{3}m_2 l_1 l_2\dot{\alpha}_1\sin(\alpha_1-\alpha_2)(\dot{\alpha}_1-\dot{\alpha}_2)\\&-\frac{1}{3}m_2 l_1 l_2\dot{\alpha}_1\dot{\alpha}_2\sin(\alpha_1-\alpha_2)-m_2 g\frac{l_2}{2}\sin\alpha_2\\&=M_2.\end{aligned} \tag{7}$$

3 The linearisation of inverted pendulum

The double inverse pendulum is described by two non-linear equations of motion. We need to linearize this non-linear system in operating point closely to steady state. Assuming small deviations we can use the following angle approximations [5]:

$$\begin{aligned}&\alpha_1\approx\alpha_2\approx 0,\\&\cos\alpha_1\approx\cos\alpha_2\approx 1,\\&\sin\alpha_1\approx\alpha_1,\\&\sin\alpha_2\approx\alpha_2,\\&\alpha_1-\alpha_2\approx 0,\\&\cos(\alpha_1-\alpha_2)\approx 1,\\&\sin(\alpha_1-\alpha_2)\approx\alpha_1-\alpha_2,\\&\dot{\alpha}_1^2\approx\dot{\alpha}_2^2\approx 0.\end{aligned} \tag{8}$$

After application of previous equations, motion equations take the form:

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$$\frac{1}{3}l_1^2\ddot{\alpha}_1(m_1+m_2)+\frac{1}{3}m_2l_1l_2\ddot{\alpha}_2 -\alpha_1\left(m_1g\frac{l_1}{2}+m_2gl_1\right)=M_1, \quad (9)$$

$$\frac{1}{3}m_2l_2^2\ddot{\alpha}_2+\frac{1}{3}m_2l_1l_2\ddot{\alpha}_1-m_2g\frac{l_2}{2}\alpha_2=M_2. \quad (10)$$

Linearized motion equations (9) and (10) we rewrite to the form of state equations [5]:

$$\dot{\mathbf{x}}=\mathbf{Ax}+\mathbf{Bu}, \quad \mathbf{y}=\mathbf{Cx}+\mathbf{Du}. \quad (11)$$

After substituting the values of parameters for the inverted pendulum:

- m_1=0,5kg,
- m_2=0,5kg,
- l_1=1m,
- l_2=1m,

form of matrices is as follows:

$$\mathbf{A}=\begin{bmatrix}0&1&0&0\\44{,}145&0&-14{,}715&0\\0&0&0&1\\-44{,}145&0&29{,}43&0\end{bmatrix},$$

$$\mathbf{B}=\begin{bmatrix}0&0\\6&-6\\0&0\\-6&12\end{bmatrix},\mathbf{C}=\begin{bmatrix}1&0&0&0\\0&0&1&0\end{bmatrix}, \quad (12)$$

$$\mathbf{D}=\begin{bmatrix}0&0\\0&0\end{bmatrix},\mathbf{u}=\begin{bmatrix}M_1\\M_2\end{bmatrix}.$$

4 The linear quadratic regulator

A state-space design approach is well suited to the control of multiple outputs as we have here. This problem can be solved using full-state feedback. The schematic of this type of control system is shown below (Figure 3) where ***K*** is a *matrix of control gains.*

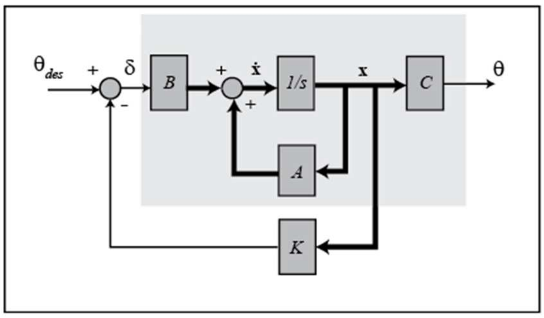

Figure 3 The schematic of full-state feedback control system [6]

The first step in designing a full-state feedback controller is to determine the open-loop poles of the system. The open-loop poles for inverted pendulum we can solve using command in programme MATLAB [6]:

- poles=eig(A),

where we work with matrix ***A***. These poles can be obtained also by solving of the characteristic equation of *transfer function*.

Based on the above mentioned matrices from equation (12) we will make the transfer functions using *Laplace transform* assuming zero initial conditions. Laplace transform is yet another operational tool for solving constant coefficients linear differential equations (Figure 4). The process of solution consists of three main steps:

- the given "hard" problem is transformed into a "simple" equation,
- this simple equation is solved by purely algebraic manipulations,
- the solution of the simple equation is transformed back to obtain the solution of the given problem.

In this way the Laplace transformation reduces the problem of solving a differential equation to an algebraic problem. The third step is made easier by tables, whose role is similar to that of integral tables in integration [7].

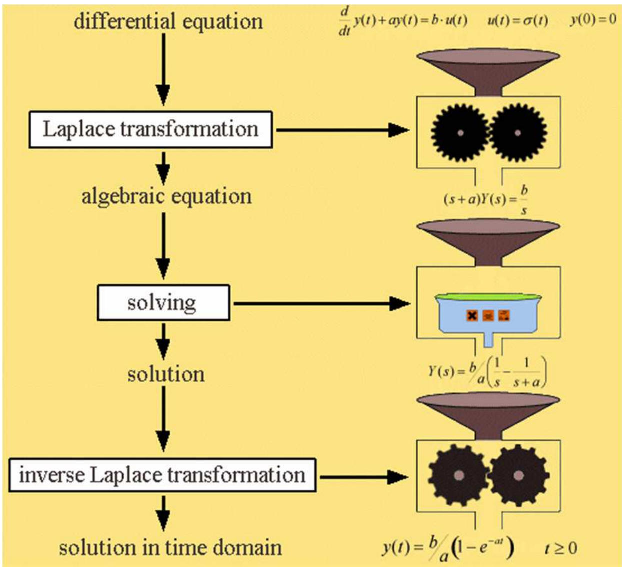

Figure 4 The schematic of Laplace and inverse Laplace transform [8]

The transfer functions of inverted pendulum created using the Laplace transform express relation between outputs α_1, α_2 and inputs M_1, M_2 and their form is:

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$$\frac{\alpha_1}{M_1}=\frac{6s^2-2{,}665.10^{-15}s-88{,}29}{s^4+2{,}665.10^{-15}s^3-73{,}57s^2-3{,}126.10^{-13}s+649{,}6}$$
$$\frac{\alpha_2}{M_1}=\frac{-6s^2+1{,}588.10^{-22}s+1{,}213.10^{-13}}{s^4+2{,}665.10^{-15}s^3-73{,}57s^2-3{,}126.10^{-13}s+649{,}6}$$
$$\frac{\alpha_1}{M_2}=\frac{-6s^2-2{,}528.10^{-14}}{s^4+2{,}665.10^{-15}s^3-73{,}57s^2-3{,}126.10^{-13}s+649{,}6} \quad (13)$$
$$\frac{\alpha_2}{M_2}=\frac{12s^2+1{,}066.10^{-14}s-264{,}9}{s^4+2{,}665.10^{-15}s^3-73{,}57s^2-3{,}126.10^{-13}s+649{,}6}$$

The poles for inverted pendulum are:

- p_1=-7,9571,
- p_2=7,9571,
- p_3=-3,2031,
- p_4=3,2031.

The two poles of the open control system are located in the right half-plane of complex variable s, from it follows that the double inverted pendulum is unstable system (Figure 5).

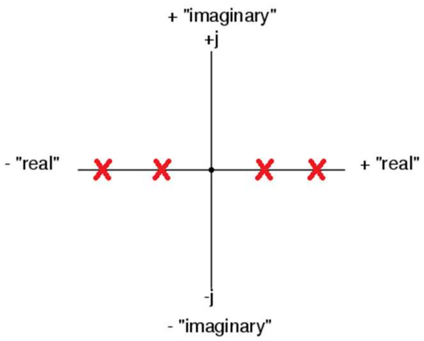

Figure 5 Two poles of the open control system

On the following figure (Figure 6) is course of the angular rotation of joints unstable inverted pendulum α_1 and α_2 when inputs M_1 and M_2 are *1Nm*.

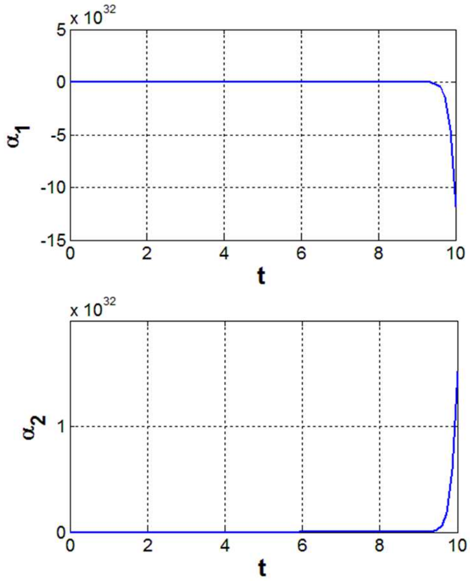

Figure 6 The course of the angular rotation of joints of unstable inverted pendulum α_1 and α_2

Before we design our controller, we will first verify that the system is *controllable*. Satisfaction of this property means that we can drive the state of the system anywhere we like in finite time (under the physical constraints of the system). For the system to be completely state controllable, the controllability matrix *CO* must have rank *n* where the rank of a matrix is the number of independent rows (or columns). The number *n* corresponds to the number of state variables of the system. The controllability of system was established using commands [6]:

- *CO=ctrb (transfer_functions),*
- *controllability=rank(CO).*

Inverted pendulum is described by four state variables and the rank of controllability matrix is four, that is, it follows that *the system is controllable.*

We will use *the linear quadratic regulation* method for determining our state-feedback control gain matrix ***K***.

Na jej výpočet opäť využijeme MATLAB pomocou príkazov [6]:

- *Q=C'*C,*
- *R=[1 0;0 1],*
- *K=lqr(A,B,Q,R).*

After determining state-feedback control gain matrix:

$$\mathbf{K}=\begin{bmatrix}14{,}0666 & 2{,}8816 & 1{,}1348 & 1{,}0010\\ -3{,}1814 & 0{,}1161 & 4{,}8456 & 0{,}9957\end{bmatrix} \quad (14)$$

We can assemble the transfer functions of stable inverse pendulum:

$$\frac{\alpha_1}{M_1}=\frac{6s^2+35{,}85s+86{,}15}{s^4+22{,}54s^3+180{,}4s^2+598{,}7s+708}$$
$$\frac{\alpha_2}{M_1}=\frac{-6s^2-4{,}179s+114{,}5}{s^4+22{,}54s^3+180{,}4s^2+598{,}7s+708}$$
$$\frac{\alpha_1}{M_2}=\frac{-6s^2-36{,}04-40{,}85}{s^4+22{,}54s^3+180{,}4s^2+598{,}7s+708} \quad (15)$$
$$\frac{\alpha_2}{M_2}=\frac{12s^2+103{,}7s+241{,}5}{s^4+22{,}54s^3+180{,}4s^2+598{,}7s+708}$$

To achieve a vertical position up was added another block *PID controller* to the block diagram. A proportional–integral–derivative controller is a control loop feedback mechanism (controller) commonly used in industrial control systems. A PID controller continuously calculates an *error value* as the difference between a desired setpoint and a measured process variable. The controller attempts to minimize the error over time by adjustment of a *control variable*, such as the position of a control valve, a damper, or the power supplied to a heating element, to a new value determined by a weighted sum:

$$u(t)=K_p e(t)+K_i\int_0^t e(\tau)d\tau+K_d\frac{de(t)}{dt} \quad (16)$$

where K_p, K_i and K_d all non-negative, denote the coefficients for the proportional, integral, and derivative

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terms, respectively (sometimes denoted P, I, and D). In this model:

- *P* accounts for present values of the error. For example, if the error is large and positive, the control output will also be large and positive.
- *I* accounts for past values of the error. For example, if the current output is not sufficiently strong, error will accumulate over time, and the controller will respond by applying a stronger action.
- *D* accounts for possible future values of the error, based on its current rate of change.

As a PID controller relies only on the measured process variable, not on knowledge of the underlying process, it is broadly applicable. By tuning the three parameters of the model, a PID controller can deal with specific process requirements. The response of the controller can be described in terms of its responsiveness to an error, the degree to which the system overshoots a setpoint, and the degree of any system oscillation. The use of the PID algorithm does not guarantee optimal control of the system or even its stability.

Some applications may require using only one or two terms to provide the appropriate system control. This is achieved by setting the other parameters to zero. A PID controller will be called a PI, PD, P or I controller in the absence of the respective control actions. PI controllers are fairly common, since derivative action is sensitive to measurement noise, whereas the absence of an integral term may prevent the system from reaching its target value [9].

The individual components P, I and D were automatically generated after running the button *Tune*, by which was designed the most suitable controller for controlling of inverted pendulum. From the transfer functions we constructed the course of the angular rotation α_1 and α_2 of the joints (Figure 7), where we can see that *the system is stable.*

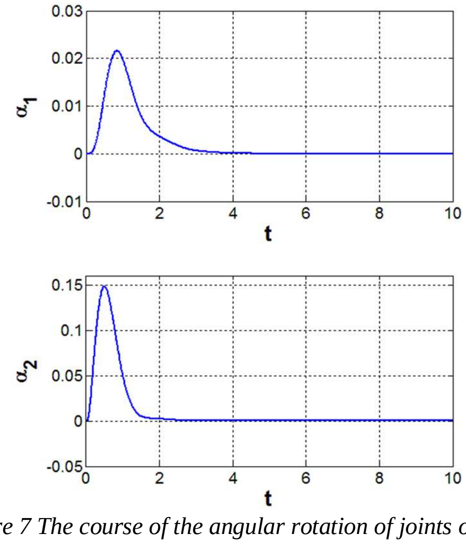

Figure 7 The course of the angular rotation of joints of stable inverted pendulum α_1 and α_2

To the block diagram (Figure 9) were finally added also DC motors with power supply 6V and their parameters are [10]:

- L=0,000121H,
- k_m=0,00449Nm/A,
- k_b=0,00448Vs/rad,
- J=2,18.10^{-5}kgm^2,
- B=9,0946Nms/rad,
- R=2,22Ω.

Using Laplace transform we created the transfer function for DC motors:

$$\frac{\alpha_{1,2}}{U} = \frac{0{,}00449}{2{,}637.10^{-9}s^2 + 0{,}001148843s + 27{,}19}. \quad (17)$$

The process of generating values of PID controller was again realized. The course the angular rotation of joints of inverted pendulum, when on input to block diagram is already placed power supply of DC motors, we can see in the figure below (Figure 8).

The simulations were carried out with zero initial conditions and deviations in individual courses were caused only due to gravity.

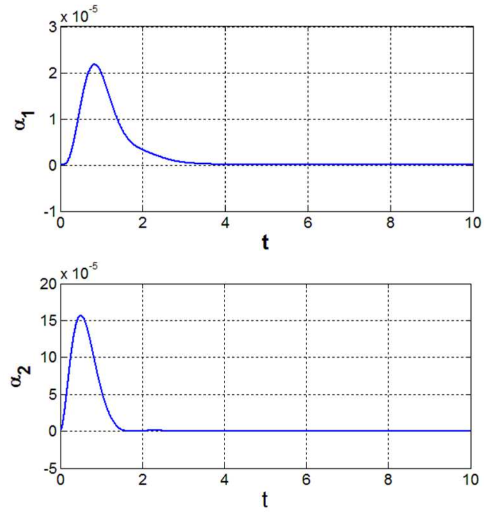

Figure 8 The course of the angular rotation of joints of stable inverted pendulum α1 and α2 in connection with DC motors

Conclusion

The article deals with the issue of stabilization of the double inverted pendulum. After the determination of the mathematical model of pendulum using Euler-Lagrange equation, we were able to linearize system. From the mathematical model transformed into the state space through MATLAB were verify the stability and controllability of system. After the determination state-feedback control gain matrix K using linear quadratic control method and individual components of PID controller were created block diagrams, which were then also simulated.

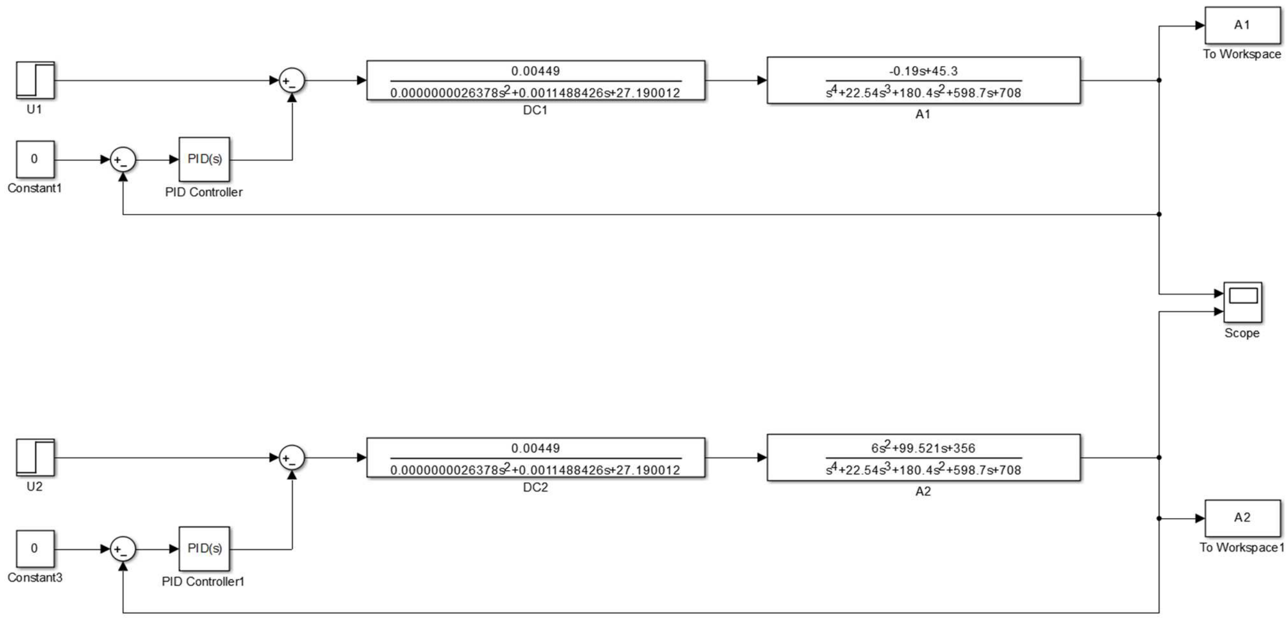

Figure 9 The complete block diagram of the double inverted pendulum expressed using a complex variables

Acknowledgement
The work has been accomplished under the research projects No. VEGA 1/0872/16 "Research of synthetic and biological inspired locomotion of mechatronic systems in rugged terrain".

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