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NEW METHOD OF THREE-POINT VIBRATION MEASUREMENTS OF TENSILE MODULUS OF THIN SAMPLES – AND ITS APPLICATION TO THE VARIETY OF SPECIMENS

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Abstract: We describe an improved method of measuring the modulus of elasticity by means of three-point bending, based on dynamic approach. This method is particularly suitable for relatively short thin specimens, and – in addition - with a wide range of shaped cross-sectional variety. In conclusion, we present a comparison of this method with the classical static one for a standard circular sample.

1 Introduction

Tensile modulus (also called elastic modulus or Young's modulus) E is a constant that describes the material´s mechanical property of stiffness and is expressed as the ratio of stress to strain for a material experiencing tensile or compressive stress.

There exist several ways for measuring this quantity. Mechanical bending phenomena belong to the most beneficial principles. The three-point bend method, for example, is particularly suited for measuring thin samples such as bars, wires, strings, fibers, stalks, and the like.

A sample of length l is firmly attached to its end points. When extinguished by the force F, the sample bends into the arc., as it is issustrated in Figure 1.

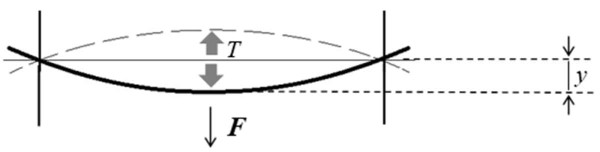

Figure 1 Arrangement principle at three-point bending experiments. A wire sample is bent by acting of force F, and vibrates with the period T when it is released

We can determine the modulus of elasticity in two ways:

1) Statically – by measuring the deflection y. After mechanical calculations, we get a relationship for the modulus of elasticity

$$E = \frac{Fl^3}{48yJ_A} \tag{1}$$

2) Dynamically - we release a tension force and we let the sample with the mass m to vibrate freely. We determine the modulus using the period T of oscillations being obtained from the relationship

$$E = \frac{4\pi^2 l^3 m}{3T^2 J_A} \tag{2}$$

The quantity of J_A means the areal moment of inertia with respect to the bending axis and is different for each shape of the sample. An overview of relations for its calculation is given in the table 1 (see section *2.3*).

The first method has a relatively wide use in practice. The second method is used less in the classical configuration due to several disadvantages: the magnitudes are fast, they have a small amplitude and they drop out quite quickly. The stabilization solution by stretching the sample into sides (like it is in the cases of guitar or violin strings, for example) is not appropriate; the magnitudes will be even faster and with less amplitude like before.

A very good and simple solution is the slowing of vibrations with damping flywheels. The magnitudes will be significantly slower, and the freewheeling of the flywheels and their permissible rotation in terms of sample bends will also create a large amplitude of vibrations and their longer duration, too. Such the counting of oscillations is even manageable "by the naked eyes", without the necessity of electronic instruments.

2 Technical and theoretical analysis of measurements

2.1 Description of the apparatus

Such a device (also known as Searle's pendulum) is illustrated in Figure 2.

It consists of three main parts: between two hinge yarns 1 are fixed horizontally the flywheels (cylindrical or prismatic) 2, they are connected by the measured sample 3; this one basically represents the element of "coupling". Usually it is in the form of wire, but it can also be a thin rod, thread or thin prismatic tape. Symmetrical deflection of the flywheels in the horizontal direction by the angle φ performs the bending oscillating movement of the sample that is reversely transmitted to the oscillating rotary motion of flywheels - and vice versa. Both parts of a pendulum – i.e. flywheels and sample - oscillate synchronously, with the same frequency and phase.

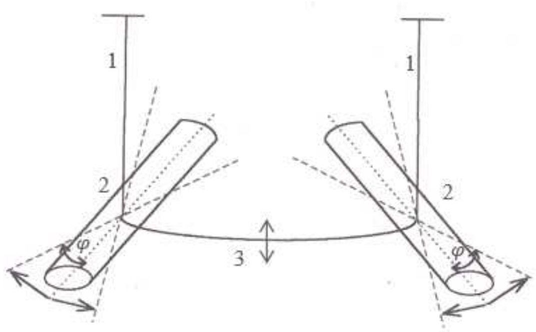

Figure 2 Three-point bending pendulum with the slowing flywheels
1 – hanging threads, 2 – cylinder flywheels, 3– measured sample (arrows indicate the direction of the oscillations)

2.2 Dynamic analysis of vibrations

The physical principle of measurement is based on the transformation of the sample bending vibrations into the oscillations of the flywheels. We proceed from the assumption that the bending moment of the wire is equal to the mechanical momentum of the force causing the rotation of the flywheels.

As we know from the mechanics, the total bending moment of the wire when deflected by the angle φ is

$$M = K \cdot \varphi = \frac{2EJ_A}{l}\varphi \,, \tag{3}$$

where K is so-called directional moment, i.e. moment of power, required for bending of wire by unit angel. The mechanical moment of the rotation force depends on the momentum of the inertia J of the flywheels; they are deflecting the same angle. We can express this process of energy transformation by the motion equation of the rotating body as

$$M = -J \cdot \varepsilon \,, \tag{4}$$

where $\varepsilon = d^2\varphi/dt^2$ is the angular acceleration of vibrations. After incorporating these statements into one relationship, we get them

$$\frac{d^2\varphi}{dt^2} + \frac{2EJ_A}{lJ}\varphi = 0 \,. \tag{5}$$

It is a known differential equation for oscillating motion; its solution is

$$\varphi = \varphi_0 \sin(\omega t + \alpha) \,, \tag{6}$$

where φ_0 is the amplitude of the oscillating motion, and α is a phase shift between zero time and moment of minimum deflection. In this relation, the new variable ω has appeared. It is the circular frequency of the oscillations that it is the result of

$$\omega = \sqrt{\frac{K}{J}} = \sqrt{\frac{2EJ_A}{lJ}} \tag{7}$$

and which is related to the oscillation period T by a known relationship

$$\omega = \frac{2\pi}{T} \tag{8}$$

We shall get the resulting relationship for E from both of these expressions

$$E = \frac{4\pi^2}{T^2} \cdot \frac{lJ}{2J_A} \tag{9}$$

The desired modulus of elasticity thus can be determined by measuring the oscillations of the sample in particular.

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In addition to the areal moment of inertia $J_{A,}$ there is acting also the moment oj inertia J of the damping flywheels. Nnow we must differentiate the type of flywheels, too. In the case of cylindrical flywheels, as well as in our picture, the moment of inertia is given by the known relationship

$$J = m\left(\frac{L^2}{12} + \frac{R^2}{4}\right). \tag{10}$$

The parameters R and L are the diameter and length of flywheels, and m means their (single) mass. In the case of square flywheels it would be

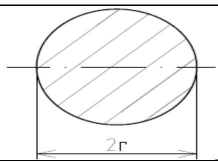
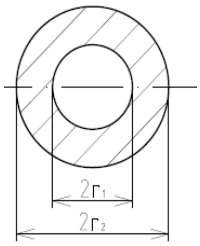
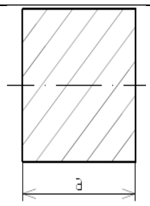
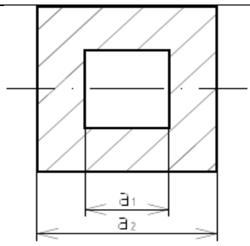
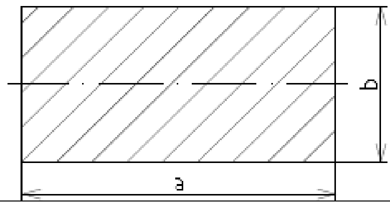
$$J = \frac{1}{12} m(A^2 + B^2), \tag{11}$$

wherein A and B are the length and the width of the prism.

2.3 *Variety of samples*

In practice, the wire samples of round shape are mostly used. However, there exist also samples with different cross-sectional shapes. An overview of the most common ones is given in tab. 1.

Table 1 Overview of possible cross-sectional shapes of wire samples

Cross-section		Areal moment of inertia
Circle - full		$J_A = \frac{1}{4}\pi r^4$
Circle - hollow		$J_A = \frac{1}{4}\pi(r_1^4 - r_2^4)$
Square - full		$J_A = \frac{1}{12}a^4$
Square - hollow		$J_A = \frac{1}{12}(a_1^4 - a_2^4)$
Rectangle		$J_A = \frac{1}{12}ab^3$

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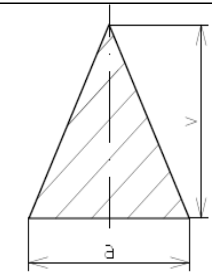
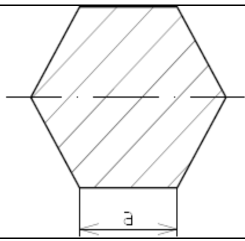
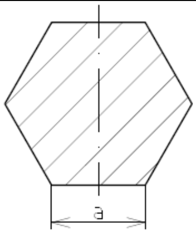
Triangle		$J_A = \frac{1}{48} v a^3$
Hexagon (top axis)		$J_A = \frac{5\sqrt{3}}{16} a^4$
Hexagon (side axis)		$J_A = \frac{\sqrt{3}}{3} a^4$

In the last column there are the relations for the calculation of the quantity J_A, which is different for each sample and which stands out in relevant relations for E.

3 Experimental procedure

3.1 Measuring assembly

Our measuring device consist of two homogeneous steel rollers in the role of flywheels, each having a mass m = 0,72 kg, a length L = 137 mm and a radius r = 14,6 mm.

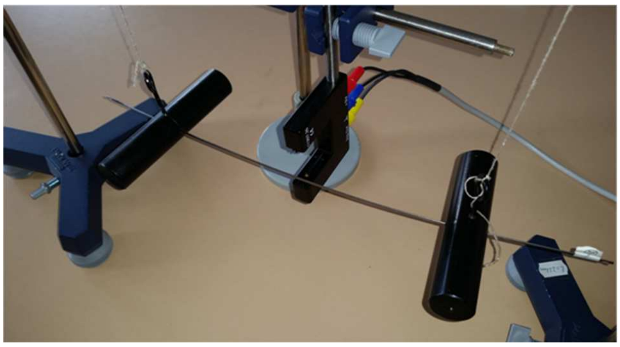


Figure 3 Experimental assembly. Vibrating wire sample crosses the infrared beam of an optical sensor (prismatic body with the shape of figure U in the centre of operation)

Size of moment of inertia of each of them, determined from the relation (10), had a value of $J = 1{,}15.10^{-3}$ kg.m^2. These were connected to one another via a wire sample being measured.

The oscillation times were scanned electronically, or by using a high-speed camera, respectively.

The photo of our device is shown in the Figure 3.

3.2 Results of measurements

We performed measurements of several samples of wires with different shapes of cross sections. All the samples had the same "active" length (i.e. the distance between the points of attachment to flywheels) l = 0,3 m. Relationships for determining the modulus of elasticity of individual samples were obtained using the presented dynamic analysis - namely the relation (9). In this relation we put the expressions for J_A for the sample in Table 1. (see the last column).

An overview of the measured samples, including the relevant geometric parameters and the measured values, is given in Table 2.

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Table 2 Parameters of samples and the results of measurements

Sample	Period of oscillation T (s)	Modulus of elasticity (measured) E_{meas} (GPa)	Table valued parameter E_{tab} (GPa)
Steel I – circle full r = 1,00 mm	0,203	210	180 – 210
Steel II – circle full r = 1,25 mm	0,135	196	180 – 210
Copper – circle full r = 1 mm	0,281	110	103 – 126
Alluminium – circle full r = 2 mm	0,126	68	65 – 70
Brass – circle full r = 1,5 mm	0,131	101	86 – 105
Steel – circle hollow r_1 = 1,1 mm; r_2 = 0,75 mm	0,195	198	180 – 210
PVC – circle full r = 1,1 mm	1,429	2,9	2,5 – 3,0
Polyamide – circle full r = 1,0 mm	2,031	2,1	1 – 2,6
Polystyrene – circle full r = 1,5 mm	0,680	3,7	3,2 – 3,5
Polypropylene – circle hollow r_1 = 1,5 mm; r_2 = 1,0 mm	1,033	2,0	1,3 – 2
Polypropylene – square full a = 2 mm	1,559	2,1	1,3 – 2
Polypropylene – square hollow a_1 = 2,0 mm; a_2 = 1,0 mm	1,740	1,8	1,3 – 2
Steel – rectangular a = 1,2 mm; b = 2,7 mm	0,130	204	180 – 210
Polypropylene – rectangular a = 1,0 mm; b = 2,7 mm	2,526	0,65	0,25 – 0,70
Polypropylene – triangle a = 2,5 mm; v = 2,2 mm	2,261	1,8	1,3 – 2
Polyamide – hexagonal a = 1,5 mm (top axis)	1,287	1,5	1 – 2,6
Polyamide – hexagonal a = 1,5 mm (side axis)	1,338	1,3	1 – 2,6
Oak wood – circle full r = 2 mm (along the fibres)	0,262	8,8	10 – 13
Corn stalk – circle hollow r_1 = 2,8 mm; r_2 = 1,0 mm	0,167	5,1	4,2 – 7
Grass stalk – circle hollow r_1 = 1,5 mm; r_2 = 1,2 mm	0,910	3,5	inaccessible value

3.3 Statistical analysis - determination of uncertainty

A "central" sample to be subjected to a more detailed analysis will be a sample of full circular cross-section.

Dynamic analysis of the process [2], [3] here gives for the measured modulus E a final relationship

$$E = \frac{8\pi l J}{r^4 T^2} \quad . \tag{12}$$

Evaluation of the corresponding uncertainty is as follows:

We must consider that the determination of the modulus of elasticity - as can be seen from equation (2) - is a function of four variables x_i; namely $E = f\,(l,J,r,T)$. In this case - in accordance with theory of measurements - the uncertainty is given by a root, containing partial derivatives with respect to all of the relevant variables and uncertainties of the following variables:

$$u_E = \sqrt{\sum_{i=1}^{n}\left(\frac{\partial E}{\partial x_i}u_{x_i}\right)^2} = \sqrt{\left(\frac{\partial E}{\partial l}u_l\right)^2 + \left(\frac{\partial E}{\partial J}u_J\right)^2 + \left(\frac{\partial E}{\partial r}u_r\right)^2 + \left(\frac{\partial E}{\partial T}u_T\right)^2} \qquad (13)$$

The relevant partial derivatives are

$$\frac{\partial E}{\partial l} = \frac{\partial \frac{8\pi l J}{r^4 T^2}}{\partial l} = \frac{8\pi J}{r^4 T^2}$$

$$\frac{\partial E}{\partial J} = \frac{\partial \frac{8\pi l J}{r^4 T^2}}{\partial J} = \frac{8\pi l}{r^4 T^2}$$

$$\frac{\partial E}{\partial r} = \frac{\partial \frac{8\pi l J}{r^4 T^2}}{\partial r} = -\frac{32\pi l J}{r^5 T^2}$$

$$\frac{\partial E}{\partial T} = \frac{\partial \frac{8\pi l J}{r^4 T^2}}{\partial T} = -\frac{16\pi l J}{r^4 T^3}$$

We felt the precision of measuring instruments for applying the uncertainties of them as the size of the smallest pieces on their scales. So:

u_L = 0,1 mm (sliding ruler)
u_r = 0,01 mm (micrometer)
u_l = 1 mm (ruler)
u_T = 0,005 s (stopwatch)
u_m = 1 g = 0,001 kg (laboratory scales).

(Quantities u_L and u_m had been used for deriving of the uncertainty u_J. Substituting into (5) gives a value of u_J = $1{,}5.10^{-5}$ kg.m^2).

The value of a numerical expression of uncertainty u_E gives a value of u_E = 14,48 GPa, which represents about 7,8 % against the size of the module being measured.

So, the final result can be written as *E* = (185,64 ± 14,48) GPa, resp. *E* = 185,64 GPa ± 7,8 %.

The value of total uncertainty is given by the sum of partial uncertainties of types A and B. As we know from the theory of measurement, the causes of uncertainty A are unknown. However, the causes of uncertainty B it is not difficult to determine, they related with an accuracy of instruments, uncertainty in the readings and air resistance against the vibrating motion. Other factors, such as a non-uniformity of wire thickness, directional moment of the hanging threads, heating the samples as a result of oscillations etc. are negligible.

3.4 Comparison with classical bending methods

For aim of comparison, we have tried to determine the modulus of elasticity using the conventional bending methods, both static and dynamic (see the introduction to this article). Calculations were performed according to relations (1) and (2). Using the same statistical procedure as in the previous case, we obtained the following results:

Static method: *E* = 191,56 GPa ± 4,3 % .

Dynamic method: *E* = 188,26 GPa ± 6,4 % .

4 Conclusion

Our equipment is less accurate than standard bending methods [5], as we can see by comparison of measurement results, mainly by means of uncertainties. But, on the other hand, it has two significant benefits:

1. The speeds of vibrations are diminished by means of flywheels, which is particularly valuable for samples with fast free oscillations. The time periods are therefore easier to measure.
2. Our system is phase-stable, it does not "tune-out" even after several tens or hundreds of oscollations. The benefit is also the possibility of measuring non-standard samples, without the risk of permanent damage.

In addition, the results are sufficiently precise, as evidenced by the fact that the measured values are well correlated with the table values.

This method can be used successfully in the wires, plastic and textile industries (investigation of elasticity of thin materials), in botany (elasticity of stalks) and the like. As so as a demonstration chapter in university textbook (section of "Vibrating Movements" or "Solid State Physics"), or a task for laboratory exercises.

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References

[1] FRIŠ, S.E., TIMOREVA, A.V.: *Kurz fysiky I*, NČSAV, 1957. (Original in Slovak)
[2] BRAWN, R.: *General Properties of Matter*, London, Butherwords, 1969.
[3] TIMOSHENKO S. , YOUNG D. H., WEAWER W.: *Vibration Problems in Engineering*, John Wiley and Sons, New York, 1974.
[4] MILJOJKOVIĆ, J., BIJELIĆ, I., VRANIČ, N., RADOVANOVIČ, N., ŽIVKOVIČ, M.: Determining Elastic Modulus of the Material by Measuring the Deflection of the Beam Loaded in Bending, *Tehnički vjesnik*, Vol. 24, No. 4, p.1227-1234, 2017.
[5] ŠTUBŇA, I., KOZÍK, T.: Factors Affecting the Accuracy of the Measurement Elastic Modulus and Mechanical Strength, *Sklář a keramik*, Vol. 1979, No. 8, p. 228-230, 1979. (Original in Czech)

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